Aileron Effectiveness

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1 Introduction

When the ailerons are deflected, the lift distribution is altered in such a way, that there is a net rolling moment causing the airplane to build up a roll rate. As soon as the airplane rolls, there is an opposing rolling moment, caused by the change in angle of attack on the wing section. Simply, it can be said that for a given aileron deflection, the roll rate builds up to the point, where these two moments are balanced.

The roll helix angle is an induced angle of attack caused due to the deflection of aileron. The helical path that the wing tip follows is often referred to as the roll helix angle. One interesting aspect is that this remains essentially constant (steady state) for a given aileron deflection.

Aileron effectiveness is a measure of how effectively ailerons are controlling the roll motion of an aircraft. In high-speed or high-stress conditions, the flexible wing structure can twist due to aileron inputs, reducing their effectiveness.

For the given jet transport aircraft, we consider an antisymmetric lift distribution case. We have calculated the aileron effectiveness taking into account the opposing moment generated by the roll helix angle and moment due to the lift distribution over the wing. The analytical calculations and the MATLAB plot of aileron effectiveness are presented in the further sections.

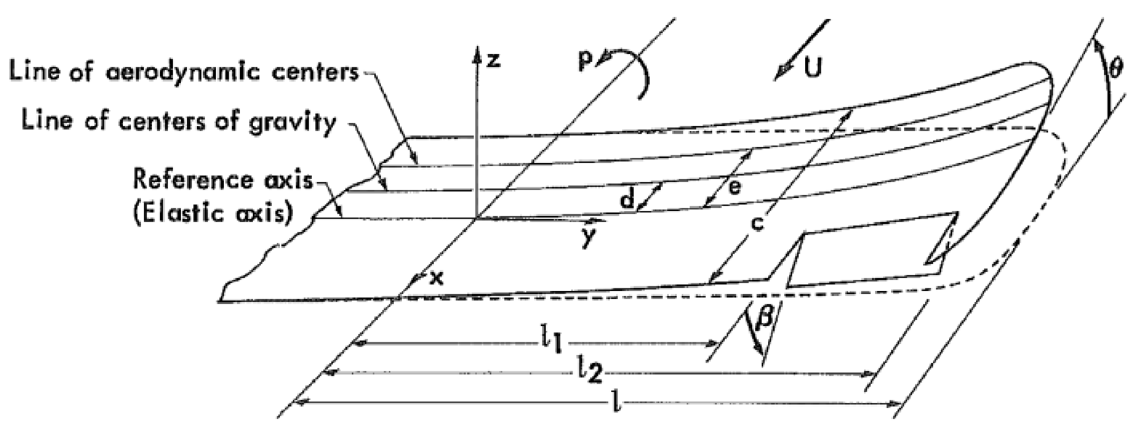

Figure 1: Elastic straight wing with deflected aileron

2 Closed Form Solution of the Aileron Effectiveness Equation

2.1 Field Equation

The field equation for the anti-symmetric lift distribution over a wing is given by,

$$\frac{d}{dy}\left[GJ\frac{d\theta}{dy}\right] + qec \cdot C_{l\alpha}\theta = -q\beta\left[ec\frac{\partial C_l^r}{\partial\beta} + c^2\frac{\partial C_{mAC}}{\partial\beta}\right]$$

$$- qec \cdot \frac{\partial C_l^r}{\partial\left(\frac{pl}{u}\right)} \cdot \left(\frac{pl}{u}\right) + y\dot{p}md \tag{1}$$

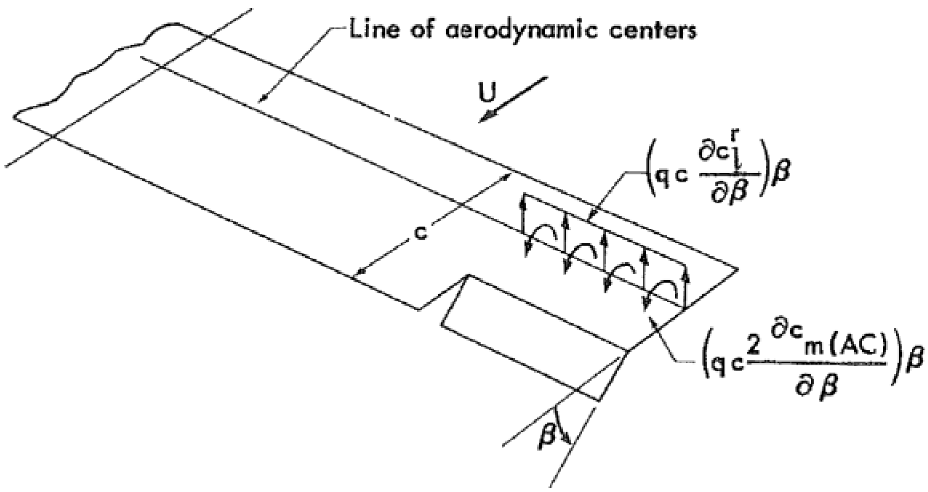

Figure 2: Forces and Moments acting on a rolling wing

On the application of strip theory and on further simplification the above equation becomes,

$$\frac{d^2\theta}{dy^2} + \lambda^2\theta = \left(\lambda^2\frac{p}{U} + k_1\dot{p}\right)y - k_2\lambda^2\beta I_a(y) \tag{2}$$

where,

$$k_1 = \frac{md}{GJ}, \quad k_2 = \frac{1}{e \cdot C_{l\alpha}}\left(e\frac{\partial C_l^r}{\partial\beta} + c\frac{\partial C_{mAC}}{\partial\beta}\right), \quad \lambda^2 = \frac{qce \cdot C_{l\alpha}}{GJ} \tag{3}$$

Implementing the boundary conditions that there is a rigid support at the root of the wing, i.e., $\theta(0) = \theta'(l) = 0$,

$$\theta(y) = \frac{1}{\lambda^2}\left(\lambda^2 \frac{p}{U} + k_1 \dot{p}\right)\left(y - \frac{\sin\lambda y}{\lambda\cos\lambda l}\right)$$

$$- k_2 \left[1_a(y)\left(1 - \cos\lambda\left(y - l_1\right)\right) - \frac{\sin\lambda\left(l - l_1\right)}{\cos\lambda l}\sin\lambda y\right]\beta \tag{4}$$

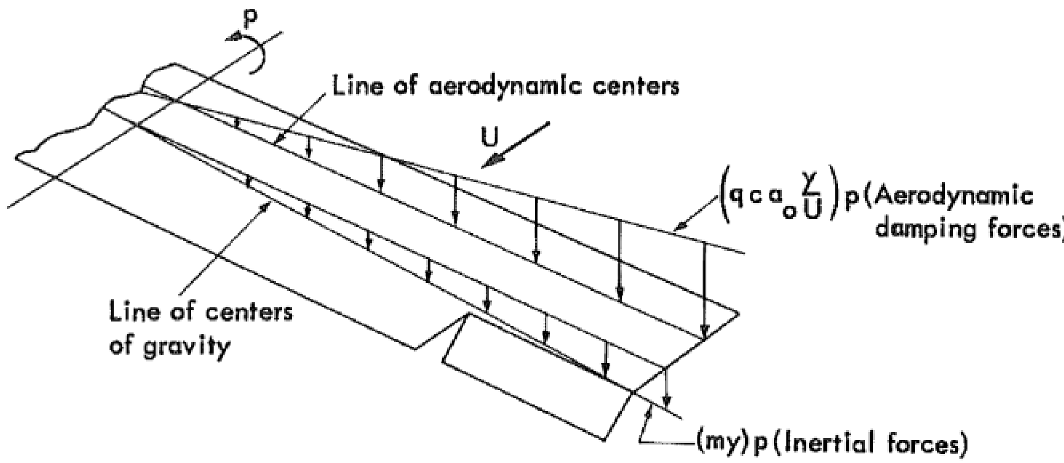

Figure 3: Damping and Inertial Forces acting on a rolling wing

The coefficients are taken as,

$$\boxed{C_1(y) = -k_2\left[I_a(y)\left(1 - \cos\lambda(y - l_1)\right) - \frac{\sin\lambda(l - l_1)}{\cos\lambda l}\sin\lambda y\right]} \tag{5}$$

$$\boxed{C_2(y) = \left(\frac{y}{l} - \frac{\sin\lambda y}{\lambda l\cos\lambda l}\right)} \tag{6}$$

2.2 Aileron Effectiveness

For steady rolling condition, p is constant, hence the roll acceleration $\dot{p}$ would be zero. Therefore,

$$T = I_{xx} \cdot \dot{p} = 0 \tag{7}$$

This can be further simplified as,

$$2q\int_0^l c \cdot C_l \cdot y \cdot dy = 0 \tag{8}$$

Or in other terms,

$$\int_0^l C_l \cdot y \cdot dy = 0 \tag{9}$$

The coefficient of lift has two components as given by the following equation,

$$C_l = C_l^r + C_l^e \tag{10}$$

where C_l^r is the rigid component of lift and C_l^e corresponds to the elastic component of lift. The rigid component can be further expressed as,

$$C_l^r(y) = \frac{\partial C_l^r(y)}{\partial \beta} \cdot \beta + \frac{\partial C_l^r(y)}{\partial \left(\frac{pl}{u}\right)} \cdot \left(\frac{pl}{u}\right) \tag{11}$$

where β is the aileron deflection and $\left(\frac{pl}{u}\right)$ is the roll helix angle. The elastic component of lift can be further expressed as,

$$C_l^e = C_{l\alpha}\left\{C_1(y)\beta + C_2(y)\left(\frac{pl}{u}\right) + C_3(y)\dot{p}\right\} \tag{12}$$

Substituting Eq. 10 - Eq. 12 in Eq. 9,

$$\int_0^l \left[\frac{\partial C_l^r(y)}{\partial \beta} \cdot \beta + \frac{\partial C_l^r(y)}{\partial \left(\frac{pl}{u}\right)} \cdot \left(\frac{pl}{u}\right) + C_{l\alpha}\left\{C_1(y)\beta + C_2(y) \cdot \left(\frac{pl}{u}\right) + C_3(y)\dot{p}\right\}\right] \cdot y \cdot dy = 0 \tag{13}$$

Rearranging the terms and substituting $\dot{p} = 0$,

$$\beta\left[\int_0^l \left\{\frac{\partial C_l^r(y)}{\partial \beta} + C_{l\alpha}C_1(y)\right\} \cdot y \cdot dy\right] = -\left(\frac{pl}{u}\right)\left[\int_0^l \left\{\frac{\partial C_l^r(y)}{\partial \left(\frac{pl}{u}\right)} + C_{l\alpha}C_2(y)\right\} \cdot y \cdot dy\right] \tag{14}$$

Rearranging the terms,

$$\beta = -\left(\frac{pl}{u}\right)\frac{\int\limits_0^l \left\{\frac{\partial C_l^r(y)}{\partial\left(\frac{pl}{u}\right)} + C_{l\alpha}C_2(y)\right\}\cdot y\cdot dy}{\int\limits_0^l \left\{\frac{\partial C_l^r(y)}{\partial\beta} + C_{l\alpha}C_1(y)\right\}\cdot y\cdot dy} \tag{15}$$

This results in the relation,

$$\boxed{\frac{\left(\frac{pl}{u}\right)}{\beta} = -\frac{\int\limits_0^l \left\{\frac{\partial C_l^r(y)}{\partial\beta} + C_{l\alpha}C_1(y)\right\}\cdot y\cdot dy}{\int\limits_0^l \left\{\frac{\partial C_l^r(y)}{\partial\left(\frac{pl}{u}\right)} + C_{l\alpha}C_2(y)\right\}\cdot y\cdot dy}} \tag{16}$$

The above equation (Eq. 16) represents the aileron effectiveness. Substituting Eq. 5 and Eq. 6 in Eq. 16,

$$\frac{\left(\frac{pl}{u}\right)}{\beta} = -\frac{\int\limits_0^l \left\{\frac{\partial C_l^r(y)}{\partial\beta} - C_{l\alpha}k_2\left[I_a(y)\left(1-\cos\lambda(y-l_1)\right) - \frac{\sin\lambda(l-l_1)}{\cos\lambda l}\sin\lambda y\right]\right\}\cdot y\cdot dy}{\int\limits_0^l \left\{\frac{\partial C_l^r(y)}{\partial\left(\frac{pl}{u}\right)} + C_{l\alpha}\left(\frac{y}{l} - \frac{\sin\lambda y}{\lambda l\cos\lambda l}\right)\right\}\cdot y\cdot dy} \tag{17}$$

we intend to derive the closed form solution of the above equation. The change in the effective angle of attack due to the rolling motion of the wing can be expressed as,

$$\alpha_{eff} = -\frac{py}{u} \tag{18}$$

This can be justified due to the introduction of the spanwise component of velocity at any location y and must be negative since as the wing rolls the effective angle of attack decreases. This in turn affects the lift coefficient,

$$C_l = C_{l\alpha}\cdot\alpha \tag{19}$$

Substituting the α_{eff},

$$C_l = C_{l\alpha} \cdot \left(-\frac{py}{u}\right) \tag{20}$$

This would be the rigid component of lift, thus this can be written as,

$$C_l^r = C_{l\alpha} \cdot \left(-\frac{py}{u}\right) \tag{21}$$

Simplifying this further and expressing it in terms of the roll helix angle we get,

$$C_l^r = -C_{l\alpha} \cdot \frac{y}{l} \cdot \left(\frac{pl}{u}\right) \tag{22}$$

Now taking the derivative with respect to the roll helix angle we obtain the relation,

$$\frac{\partial C_l^r(y)}{\partial \left(\frac{pl}{u}\right)} = -C_{l\alpha} \cdot \frac{y}{l} \tag{23}$$

Thus Eq. 17 can be reduced to,

$$\frac{\left(\frac{pl}{u}\right)}{\beta} = -\frac{\int_0^l \left\{ \frac{\partial C_l^r(y)}{\partial \beta} - C_{l\alpha} k_2 \left[I_a(y) \left(1 - \cos\lambda(y - l_1)\right) - \frac{\sin\lambda(l - l_1)}{\cos\lambda l} \sin\lambda y \right] \right\} \cdot y \cdot dy}{\int_0^l \left\{ -C_{l\alpha} \cdot \frac{y}{l} + C_{l\alpha} \left(\frac{y}{l} - \frac{\sin\lambda y}{\lambda l \cos\lambda l} \right) \right\} \cdot y \cdot dy} \tag{24}$$

Simplifying this further,

$$\frac{\left(\frac{pl}{u}\right)}{\beta} = \frac{\int_0^l \left\{ \frac{\partial C_l^r(y)}{\partial \beta} - C_{l\alpha} k_2 \left[I_a(y) \left(1 - \cos\lambda(y - l_1)\right) - \frac{\sin\lambda(l - l_1)}{\cos\lambda l} \sin\lambda y \right] \right\} \cdot y \cdot dy}{\int_0^l \left\{ C_{l\alpha} \left(\frac{\sin\lambda y}{\lambda l \cos\lambda l} \right) \right\} \cdot y \cdot dy} \tag{25}$$

The numerator of Eq. 25 is given by,

$$\int_0^l \left\{ \frac{\partial C_l^r(y)}{\partial \beta} - C_{l\alpha} \cdot k_2 \left[I_a(y) \left(1 - \cos\lambda(y - l_1)\right) - \frac{\sin\lambda(l - l_1)}{\cos\lambda l} \sin\lambda y \right] \right\} \cdot y \cdot dy \tag{26}$$

On simplification,

$$\int_0^l \left\{ \frac{\partial C_l^r(y) \cdot y}{\partial \beta} - C_{l\alpha} \cdot y \cdot k_2 \left[I_a(y)\left(1 - \cos\lambda(y - l_1)\right) - \frac{\sin\lambda(l - l_1)}{\cos\lambda l} \sin\lambda y \right] \right\} \cdot dy \tag{27}$$

Splitting the integral,

$$\int_0^l C_{l\alpha} \cdot y \cdot k_2 \left[\frac{\sin\lambda(l - l_1)}{\cos\lambda l} \sin\lambda y \right] \cdot dy + \int_{l_1}^l \frac{\partial C_l^r(y) \cdot y}{\partial \beta}$$
$$- \int_{l_1}^l C_{l\alpha} \cdot y \cdot k_2 \left[I_a(y)\left(1 - \cos\lambda(y - l_1)\right)\right] \cdot dy \tag{28}$$

On further simplification and substitution of $I_a(y) = 1$ in the limits of $l \to l_1$,

$$C_{l\alpha} \cdot k_2 \cdot \left\{ \int_0^l \left[\frac{\sin\lambda(l - l_1)}{\cos\lambda l} \cdot y \sin\lambda y \right] dy + \frac{1}{C_{l\alpha} \cdot k_2} \int_{l_1}^l \frac{\partial C_l^r(y)}{\partial \beta} \cdot y \cdot dy \right.$$
$$\left. - \int_{l_1}^l \left[y - y\cos\lambda(y - l_1) \right] dy \right\} \tag{29}$$

On integrating,

$$C_{l\alpha} \cdot k_2 \cdot \left\{ \frac{\sin\lambda(l - l_1)}{\cos\lambda l} \left[-y \cdot \frac{\cos\lambda y}{\lambda} + \frac{\sin\lambda y}{\lambda^2} \right]_0^l + \frac{1}{C_{l\alpha} \cdot k_2} \cdot \frac{\partial C_l^r(y)}{\partial \beta} \cdot \left[\frac{y^2}{2} \right]_{l_1}^l \right.$$
$$\left. + \left[-\frac{y^2}{2} + y \cdot \frac{\sin\lambda(y - l_1)}{\lambda} + \frac{\cos\lambda(y - l_1)}{\lambda^2} \right]_{l_1}^l \right\} \tag{30}$$

Substituting the limits,

$$C_{l\alpha} \cdot k_2 \cdot \left\{ \frac{\sin\lambda(l - l_1)}{\cos\lambda l} \left[-l \cdot \frac{\cos\lambda l}{\lambda} + \frac{\sin\lambda l}{\lambda^2} \right] + \frac{1}{C_{l\alpha} \cdot k_2} \cdot \frac{\partial C_l^r(y)}{\partial \beta} \cdot \left[\frac{l^2}{2} - \frac{l_1^2}{2} \right] \right.$$
$$\left. + \left[-\frac{l^2}{2} + l \cdot \frac{\sin\lambda(l - l_1)}{\lambda} + \frac{\cos\lambda(l - l_1)}{\lambda^2} \right] - \left[-\frac{l_1^2}{2} + \frac{1}{\lambda^2} \right] \right\} \tag{31}$$

On further simplification the numerator results in the following expression,

$$\frac{C_{l\alpha}\cdot k_2}{\lambda^2}\left[\frac{\cos\lambda l_1}{\cos\lambda l}-\lambda^2\cdot\frac{l^2-l_1^2}{2}-1\right]+\frac{\partial C_l^r(y)}{\partial\beta}\cdot\frac{l^2-l_1^2}{2} \tag{32}$$

Substituting k_2 we obtain the following relation,

$$\frac{C_{l\alpha}}{\lambda^2}\cdot\left\{\left[\left(\frac{\cos\lambda l_1}{\cos\lambda l}-1\right)\frac{1}{C_{l\alpha}}\frac{\partial C_l^r(y)}{\partial\beta}\right]+\left(\frac{\cos\lambda l_1}{\cos\lambda l}-1-\lambda^2\frac{l^2-l_1^2}{2}\right)\frac{c}{e\cdot C_{l\alpha}}\frac{\partial C_{mAC}}{\partial\beta}\right\} \tag{33}$$

The denominator of Eq. 25 is given by,

$$\int_0^l\left\{C_{l\alpha}\left(\frac{\sin\lambda y}{\lambda l\cos\lambda l}\right)\right\}\cdot y\cdot dy \tag{34}$$

Simplifying,

$$\frac{C_{l\alpha}}{\lambda l\cos\lambda l}\int_0^l y\sin\lambda y\cdot dy \tag{35}$$

On integrating,

$$\frac{C_{l\alpha}}{\lambda l\cos\lambda l}\left[-y\cdot\frac{\cos\lambda y}{\lambda}+\frac{\sin\lambda y}{\lambda^2}\right]_0^l \tag{36}$$

Substituting the limits,

$$\frac{C_{l\alpha}}{\lambda l\cos\lambda l}\left[-l\cdot\frac{\cos\lambda l}{\lambda}+\frac{\sin\lambda l}{\lambda^2}\right] \tag{37}$$

Simplifying we get the following relation,

$$\frac{C_{l\alpha}}{\lambda^2}\left(\frac{\tan\lambda l}{\lambda l}-1\right) \tag{38}$$

Substituting Eq. 33 and Eq. 38 in Eq. 25 we get,

$$\frac{\left(\frac{pl}{u}\right)}{\beta}=\frac{\frac{C_{l\alpha}}{\lambda^2}\left\{\left[\left(\frac{\cos\lambda l_1}{\cos\lambda l}-1\right)\frac{1}{C_{l\alpha}}\frac{\partial C_l^r(y)}{\partial\beta}\right]+\left(\frac{\cos\lambda l_1}{\cos\lambda l}-1-\lambda^2\frac{l^2-l_1^2}{2}\right)\frac{c}{e\cdot C_{l\alpha}}\frac{\partial C_{mAC}}{\partial\beta}\right\}}{\frac{C_{l\alpha}}{\lambda^2}\left(\frac{\tan\lambda l}{\lambda l}-1\right)} \tag{39}$$

The final form of the equation is given by,

$$\frac{\left(\frac{pl}{u}\right)}{\beta} = \frac{\left[\left(\frac{\cos\lambda l_1}{\cos\lambda l} - 1\right)\frac{1}{C_{l\alpha}}\frac{\partial C_l^r(y)}{\partial\beta}\right] + \left(\frac{\cos\lambda l_1}{\cos\lambda l} - 1 - \lambda^2\frac{l^2 - l_1^2}{2}\right)\frac{c}{e\cdot C_{l\alpha}}\frac{\partial C_{mAC}}{\partial\beta}}{\left(\frac{\tan\lambda l}{\lambda l} - 1\right)} \tag{40}$$

The above equation is the closed form solution of the aileron effectiveness equation. The following figure shows the variation of the roll helix angle with λl.

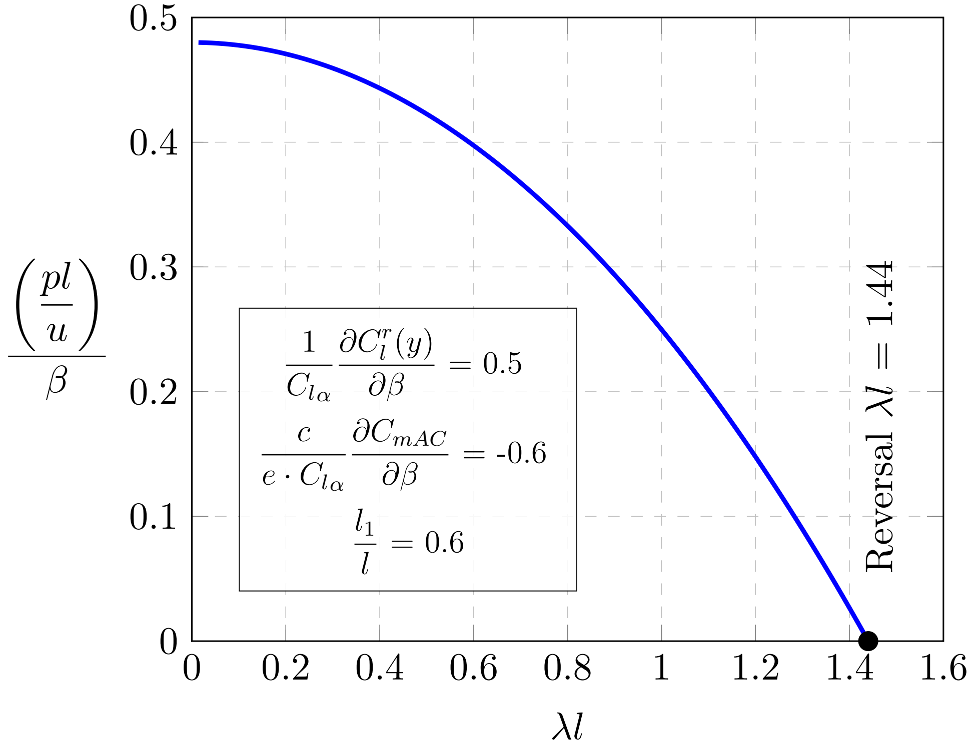

Figure 4: Aileron effectiveness of uniform wing as a function of λl

3 Integral Form of the Aileron Effectiveness Equation

3.1 Field Equation

The field equation for the anti-symmetric lift distribution over a wing is given by,

$$\frac{d}{dy}\left[GJ\frac{d\theta}{dy}\right] + qec\cdot C_{l\alpha}\theta = -q\beta\left[ec\frac{\partial C_l^r(y)}{\partial\beta} + c^2\frac{\partial C_{mAC}}{\partial\beta}\right]$$

$$- qec\cdot\frac{\partial C_l^r(y)}{\partial\left(\frac{pl}{u}\right)}\cdot\left(\frac{pl}{u}\right) + y\dot{p}md \tag{41}$$

3.2 Aileron Effectiveness

The solution of the above equation is given by,

$$\theta(y) = q\int_0^l C^{\theta\theta}(y,\eta)\cdot e\cdot c\cdot c_l^e\cdot d\eta + f_a(y) \tag{42}$$

$f_a(y)$ is given by,

$$f_a(y) = \int_0^l C^{\theta\theta}(y,\eta)\left[qec\left(\frac{\partial C_l^r(y)}{\partial\beta}\beta + \frac{\partial C_l^r}{\partial\left(\frac{pl}{u}\right)}\left(\frac{pl}{u}\right)\right) + qc^2\frac{\partial C_{m\mathrm{AC}}}{\partial\beta}\beta - m\eta\dot{p}d\right]d\eta \tag{43}$$

$\theta(y)$ is given by,

$$\theta(y) = q\int_0^l C^{\theta\theta}(y,\eta)e\cdot c\cdot C_l^e d\eta + q\beta\int_0^l C^{\theta\theta}(y,\eta)\left(e\cdot\frac{\partial C_l^r}{\partial\beta} + c\cdot\frac{\partial C_{m\mathrm{AC}}}{\partial\beta}\right)cd\eta$$

$$\left(\frac{pl}{u}\right)\int_0^l C^{\theta\theta}(y,\eta)\frac{\partial C_l^r}{\partial\left(\frac{pl}{u}\right)}c\cdot e\cdot d\eta \tag{44}$$

Assuming that the aerodynamic and elastic axes coincide, i.e., $e = 0$

$$\frac{\theta(y)}{\beta} = q\int_0^l C^{\theta\theta}(y,\eta)\cdot\frac{\partial C_{mAC}}{\partial\beta}\cdot c^2 d\eta \tag{45}$$

The antisymmetrical twist distribution can be given as,

$$\theta(y) = C_1(y)\cdot\beta + C_2(y)\cdot\frac{pl}{u} + C_3(y)\cdot\dot{p} \tag{46}$$

The coefficients are given by,

$$C_1(y) = q\int_0^l C^{\theta\theta}(y,\eta)\cdot\frac{\partial C_{m\mathrm{AC}}}{\partial\beta}\cdot c^2 d\eta, \quad C_2(y) = 0 \tag{47}$$

$C_3(y)$ is neglected since $\dot{p} = 0$, given a steady roll rate. Substuting the above values in Eq. 16 we get,

$$\frac{\left(\frac{pl}{u}\right)}{\beta} = -\frac{\int_0^l \left\{ \frac{\partial C_l^r(y)}{\partial \beta} + C_{l\alpha} \left[q \int_0^l C^{\theta\theta}(y,\eta) \cdot \frac{\partial C_{m\mathrm{AC}}}{\partial \beta} \cdot c^2 d\eta \right] \right\} \cdot y \cdot dy}{\int_0^l \frac{\partial C_l^r(y)}{\partial \left(\frac{pl}{u}\right)} \cdot y \cdot dy} \tag{48}$$

Equating the numerator to zero, we obtain the reversal dynamic pressure,

$$q_R = -\frac{\int_0^l \frac{\partial C_l^r(y)}{\partial \beta} \cdot y dy}{C_{l\alpha} \left\{ \int_0^l \left[\int_0^l C^{\theta\theta}(y,\eta) \cdot \frac{\partial C_{m\mathrm{AC}}}{\partial \beta} \cdot c^2 d\eta \right] \cdot y \cdot dy \right\}} \tag{49}$$

In the problem assigned, since the values of $C^{\theta\theta}$ are not known at different spanwise locations, it is therefore not possible to calculate the reversal dynamic pressure and hence its variation with the Mach number.

4 Conclusion

The aileron effectiveness is crucial to understanding the roll control dynamics of an aircraft, especially in the context of flexible elastic wings. The aileron effectiveness was analytically derived and the polar plotted using MATLAB indicates the variation of the control authority with λl.

5 Appendix: MATLAB Code

```
A=0.5; %(1/a0)*(dCl/dBeta)
B= -0.6; % (c/a0*e)*(dCmac/dBeta)
l_ratio= 0.6;%(l1/l)
x= linspace(0,1.4,100)  % 'x' is Lambda*l

for i =1:numel(x)
% Define the equation
numerator = ((cos(0.6*x(i)) / cos(x(i))) - 1) * A + ...
    ((cos(0.6*x(i)) / cos(x(i))) - 1 - (x(i)^2*(1 - 0.36) / 2))*B

denominator = ((tan(x(i)) / x(i)) - 1);

result(i) = numerator / denominator;
end
% Display result
plot(x,result)
```